CONCERNING THE TRACTRIX OF A CURVE, WITH PLANIMETRIC APPLICATION.

By Derrick N. Lehmer.

1. We define the *l-Tractrix of a curve* as follows:

While a point (x,y) follows a given curve, a second point (ξ,η) moves directly toward it, or directly from it, keeping at a constant distance l from the first point. We will call the path of the second point an l-Tractrix of the given curve.

If $f(x,y)=0$ be the equation of the given curve, the equation of the l-Tractrix may be obtained by eliminating x and y from the three equations

$$\begin{array}{ll} \text{I} & f(x,y)=0, \\ \text{II} & (x-\xi)^2+(y-\eta)^2=l^2, \\ \text{III} & (y-\eta)=\dfrac{d\eta}{d\xi}(x-\xi). \end{array}$$

In fact, II gives the condition that the two points be at a constant distance apart, while III is the condition that the point (x,y) should lie on the tangent to the l-Tractrix at the point (ξ,η). The actual elimination in any special case may be a matter of great difficulty.

It is evident that a curve may have an infinite number of l-Tractrices depending on the choice of the initial point of the l-Tractrix. The l-Tractrix, on the other hand, has but one base curve, if the sign of l is taken into account. The base curve is, in fact, obtained by measuring a constant distant l from the point of tangency along each tangent to the Tractrix.

2. In what follows we shall have occasion to consider areas traced out by a point which describes a closed contour. The analytical formula for such an area is

$$\int_c \left(x\frac{dy}{dt}-y\frac{dx}{dt}\right)dt$$

where x and y, — the coordinates of the moving point, — are functions of the independent variable t the time; the integral being taken around the curve so as to keep the area on the left. If the point moves in the opposite direction the area is to be taken *negatively*. It may easily happen that the positive and negative portions of an area may cancel, as in the Lemniscate of Bernoulli. If the contour is described n times the area is multiplied by n.

We shall also consider areas swept out by a revolving line. The sign of the area will depend on the direction of rotation of the line, being positive when in the direction of increasing angles.

3. THEOREM. *If a curve and its l-Tractrix are both closed, the area of the curve is equal to the area of the l-Tractrix plus k times the area of a circle of radius l, where k is the number of complete revolutions made by a tangent line in going around the l-Tractrix.*

We may replace condition number II of §1 by the two

$$x = \xi + l\cos\theta,$$
$$y = \eta + l\sin\theta,$$

where θ is the inclination to the axis of x of the line joining the two points. From these we get easily, taking the time t as independent variable,

$$(1) \qquad \left(x\frac{dy}{dt} - y\frac{dx}{dt}\right) = \left(\xi\frac{d\eta}{dt} - \eta\frac{d\xi}{dt}\right)$$
$$+ l\left(\xi\cos\theta\frac{d\theta}{dt} + \eta\sin\theta\frac{d\theta}{dt} - \sin\theta\frac{d\xi}{dt} + \cos\theta\frac{d\eta}{dt}\right) + l^2\frac{d\theta}{dt}.$$

Now it is easily verified that the second term on the right is equal to

$$l\left[\frac{d}{dt}(\xi\sin\theta - \eta\cos\theta) - 2\left(\frac{d\xi}{dt}\sin\theta - \frac{d\eta}{dt}\cos\theta\right)\right].$$

But from condition III of §1,

$$\frac{d\eta}{d\xi} = \frac{y-\eta}{x-\xi} = \frac{\sin\theta}{\cos\theta};$$

so that

$$\frac{d\eta}{dt}\cos\theta - \frac{d\xi}{dt}\sin\theta = 0,$$

and the equation (1) becomes

$$(2)\ \left(x\frac{dy}{dt} - y\frac{dx}{dt}\right) = \left(\xi\frac{d\eta}{dt} - \eta\frac{d\xi}{dt}\right) + l\frac{d}{dt}(\xi\sin\theta - \eta\cos\theta) + l^2\frac{d\theta}{dt}.$$

We may choose our origin at the initial point (x,y) and let the axis of x coincide with the initial position of the line between (x,y) and (ξ,η). We have then the initial values:

$$\xi = l,$$
$$\eta = 0,$$
$$\theta = 0.$$

Now when the point (x,y) moves around the given curve, the integral

$$\int \left(x\frac{dy}{dt} - \eta\frac{dx}{dt}\right) dt$$

becomes equal to $2A$, where A is the area enclosed by the curve.

At the same time the integral

$$\int \left(\xi\frac{d\eta}{dt} - \eta\frac{d\xi}{dt}\right) dt$$

has taken the value $2T$, where T is the area of the l-Tractrix taken with proper sign. The final values of ξ and η are the same as the initial, while θ has changed to $2k\,\pi$. Also at both limits

$$\xi \sin\theta - \eta\cos\theta = 0,$$

and we have

$$2A = 2T + 2k\,\pi\,l^2,$$

or

$$A = T + k\,\pi\,l^2, \qquad \text{Q. E. D.}$$

4. As an example of the above theorem we may take for our tractrix a regular triangle. (See Fig. 1.)

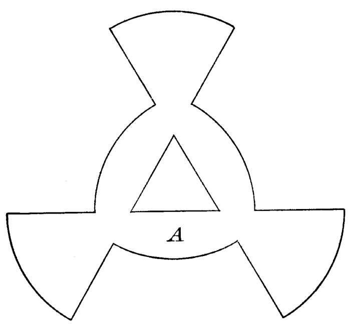

FIG. 1.

If we call Δ the area of the triangle, it is easily seen that

$$A = -\,2\,\Delta + \pi\,l^2.$$

Now in order that the outer curve be closed, it is seen to be necessary to traverse the contour of the l-Tractrix *twice*. It is further seen that the area of the l-Tractrix is traced *negatively*. Hence

$$T = -\,2\,\Delta$$

and so

$$A = T + \pi\,l^2.$$

5. The theorem of § 3 may be proved by the methods of infinitesimal geometry as follows:

Let a tangent line of fixed length l measured from the point of tangency, move along a curve. The area swept out is equal to $l^2\,\theta/2$, where θ is the difference in direction between the initial and final positions of the tangent line.

In fact the element of area is a triangle two sides of which are l and $l+ds$ while the included angle is $d\theta$. Dropping infinitesimals of the second order, the area is $l^2\,d\theta/2$, and integrating, the area is $l^2\,\theta/2$.

Otherwise, we may divide the curve into n equal parts and allow the tangent line to move along the broken line formed by joining the n points of division. The area generated is $l^2\,\theta'/2$ where θ' is the angle between the first and final chords. By increasing n indefinitely the theorem follows.

In this theorem the sign of the area is always to be considered. As an example we may take the well-known equitangential curve. The total change in the direction of the tangent line is $\pi/2$. Therefore the area bounded by the two axes and the curve is $\pi\, l^2/4$ which is a known theorem of the calculus.

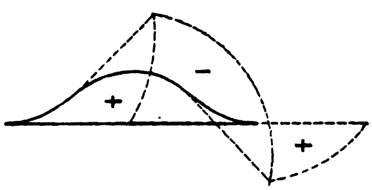

FIG. 2.

It may easily happen that the area swept out is zero as in fig. 2. Here $\theta = 0$ and the positive and negative areas exactly cancel.

Let now the curve be closed. If the tangent line makes k complete revolutions in going around it, the area generated will be $k\,\pi\,l^2$. Now this curve is the l-Tractrix of the curve generated by the extremity of the moving tangent. The areas of these two curves differ then by $k\,\pi\,l^2$ as in the theorem of § 3.

In a curve such as the cardioid it will be observed that k is not a whole number, but at the same time the curve generated by the extremity of the moving tangent is *not closed*, so that the theorem of § 3 does not apply. By tracing the curve twice however this difficulty is removed.

6. The theorem of § 3 may be made the basis of the theory of a planimeter of recent invention* which consists essentially of a bar of length l, carrying a tracing point at one end and a knife-edge at the other. (A pocket knife, with a blade at each end, opened up until the points are at a distance l apart, answers very well.) As the tracing point moves along a curve, the knife-edge traces out an l-Tractrix of that curve. The method of using the instrument is as follows: Start with the tracing point at the centre of gravity of the area to be measured. Proceed along some definite line to the contour.

* See Prytz, *Engineering*, Vol. 57, p. 813. Editorial, *Engineering*, Vol. 57, pp. 687, 725. Hill, *Phil. Mag.*, June 22, 1894.

Trace the contour and return along the same line to the starting point. Measure now the distance between the initial and final positions of the knife-edge. Multiply this distance by l. The result is the required area.

This extraordinary rule does not always give good results. The feature of starting at the centre of gravity of the area is non-essential as we shall see. Writers on the theory of the instrument have occupied themselves chiefly with the error introduced by starting at a point not exactly at the centre of gravity. Captain Prytz — who seems to be the inventor of the instrument — discusses it by means of infinite series, as does also Mr. Hill. The editorials cited in *Engineering* treat it by geometrical methods. In none of the discussions, however, can one get a definite idea of the magnitude and sign of the error which may arise in the use of the instrument, or how to better the approximation in any particular case.

7. The theorem of § 3 is not immediately applicable. The tractrix of a given closed curve is not necessarily itself a closed curve. We meet this difficulty as follows (see fig. 3):

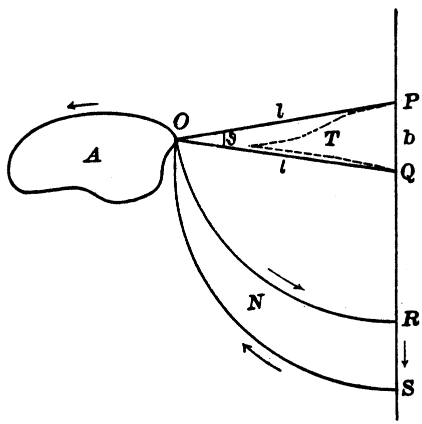

FIG. 3.

Suppose that while the tracing point starting from a point O on the boundary of A describes the contour of A in a positive sense, the knife-edge moves along the l-tractrix from Q to P. Join P and Q. With centre P and radius l describe the arc OR. With centre Q and radius l describe the arc OS. (R and S are on the line PQ.)

If the tracing point describes the contour $ORSO$ the knife-edge will return from P to R and the tractrix is now a closed curve. Our theorem then gives

$$A + N = T + k\pi l^2,$$

where all the areas are to be taken with their proper signs. Thus in the figure A is positive while N and T are negative, k being zero.

It is an easy matter to show that the area N is numerically equal to

$$\frac{l^2}{2}(\theta + \sin\theta),$$

where θ is the angle Q O P. We have then in general

$$A = T - \frac{l^2}{2}(\theta + \sin\theta) + k\pi l^2.$$

As an example we may take a circle of radius l and start with the tracing point at the centre O with the knife-edge at Q on the circumference.

Move now the tracing point from O to P to bring the knife-edge to the centre O. Describe the contour and return to O. The knife-edge moves back to Q, and the rule given in §6 would make the area zero. The above formula gives $A = \pi l^2$ since we have $\theta = 0$, $T = 0$, and $k = 1$.

8. We can make θ very small by increasing l. For small values of θ we may put $\sin\theta = \theta$, and the value $\frac{l^2}{2}(\theta + \sin\theta)$ is a little smaller than $l^2\theta$. But $l^2\theta = l \cdot l\theta$ which is a little larger than lb where b is the chord PQ. For $\theta \leqq 30°$ the error introduced by putting lb for $\frac{l^2}{2}(\theta + \sin\theta)$ is less than two parts in one thousand.

Also for values of l which are large compared with the longest line of the area to be measured, the line l will not turn a complete revolution. So k will be zero and we shall have

$$A = T - lb.$$

In starting from the centre of gravity it may happen that the area T is made up of positive and negative portions which very nearly cancel each other. In that case lb is a good approximation to the area A, and the rule given in § 6 is justified.

It is safer to use the formula

$$A = T - bl,$$

actually computing the area T. For large values of l the l-Tractrix is practically a rectilinear figure whose area is easily computed. In fact, by starting on the contour with the tracing point, instead of at the centre of gravity, we may often get a tractrix which differs very little from a triangle whose base is b. The height h may be measured and the area is

$$A = b(\tfrac{1}{2}h - l).$$

If the instrument were made to draw the tractrix one could draw the triangle and judge with considerable accuracy the magnitude and sign of his error.

Of course other errors enter, due to the drifting of the knife-edge; difficulty in tracing the contour; difficulty of measuring l and b, etc. With a little care, however, surprisingly accurate results may be obtained.

UNIVERSITY OF CHICAGO, MAY, 1899.